DESCRIPTION OF INTERNAL FRICTION IN THE MEMORY THEORY OF ELASTICITY USING KERNELS WITH A WEAK SINGULARITY

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Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 8, No. 4, pp. 147-151, 1967

In many materials the stress σ (deformation ε) at a given instant of time depends in a complex way on the entire previous history of deformation (state of stress). These properties are taken into account by the Boltzmann-Volterra memory theory of elasticity, which in the case of a uniaxial state of stress is based on the two integral equations [1]

$$\sigma(t) = E_\infty\left[\varepsilon(t) - \varkappa \int_{-\infty}^{t} f(t-t')\,\varepsilon(t')\,dt'\right], \tag{1}$$

$$\varepsilon(t) = J_\infty\left[\sigma(t) + \varkappa \int_{-\infty}^{t} \varphi(t-t')\,\sigma(t')\,dt'\right]. \tag{2}$$

Here, $f(t-t')$ is the relaxation kernel, and its resolvent $\varphi(t-t')$ is the aftereffect kernel; E_∞ is the modulus of elasticity, J_∞ is the compliance at a time when none of the relaxation and aftereffect processes has yet been realized, and $\varkappa$ is a coefficient that depends on the specific form of the kernels.

The basic problem of the memory theory of elasticity is to determine the form of the relaxation and aftereffect kernels. For this purpose it is customary to use exponential functions or discrete and continuous spectra composed of such functions [2]. These kernels permit a quite accurate description of many properties of actual bodies and, in particular, account for the temperature-frequency dependence of internal friction of the relaxation type [3]. However, the exponential kernels do not have the singularity at $t - t' = 0$ that is observed experimentally in static tests [4]. In this connection it is worthwhile to investigate some typical examples to discover how this singularity affects the dissipative properties of the medium.

In calculating the internal friction it is a matter of indifference whether we specify the relaxation kernel or the aftereffect kernel, since in the general form they are related by a simple expression which is easily established by rewriting Eqs. (1) and (2) in transform space

$$\sigma_{**} = E_\infty[1-\varkappa f_*(p)]\,\varepsilon_{**}, \qquad \varepsilon_{**} = J_\infty[1+\varkappa\varphi_*(p)]\,\sigma_{**}. \tag{3}$$

Here, a single asterisk denotes the unilateral and a double asterisk the bilateral Laplace transform. Equations (3) yield the following relations between the Laplace transforms of the aftereffect and relaxation kernels

$$f_*(p) = \varphi_*(p)\,[1+\varkappa\varphi_*(p)]^{-1}, \quad \varphi_*(p) = f_*(p)\,[1-\varkappa f_*(p)]^{-1}. \tag{4}$$

In transform space the Boltzmann-Volterra equations are analogous to Hooke's law, the only difference being that the elastic constants are a function of the complex parameter p of the unilateral Laplace transformation.

In order to investigate elastic-memory media in periodic deformation it is sufficient to rewrite the elastic constant of Eqs. (3) in Fourier space $p \to i\omega$

$$E = E_\infty\,[1-\varkappa f_*(i\omega)] = E' + iE'',$$
$$J = J_\infty\,[1+\varkappa\varphi_*(i\omega)] = J' - iJ''. \tag{5}$$

Thus, the elastic modulus and compliance (5) are complex numbers. After multiplication by ε and σ, respectively, the real components $\varepsilon E'$ and $\sigma J'$ are those parts of the stress and strain that vary in phase, while the imaginary components $\varepsilon E''$ and $\sigma J''$ have a relative phase shift of 90°. The presence of phase-shifted components determines the dissipative processes in elastic-memory media. The energy losses (internal friction) are given by the mechanical loss factor

$$\operatorname{tg}\delta = E''/E' = J''/J'. \tag{6}$$

The calculations that follow have been made for the complex compliance.

1. In accordance with Eqs. (5)-(6) with $\varkappa = \Delta E/E_\infty\tau_\varepsilon = \Delta J/J_\infty\tau_\sigma$ the exponential kernels $f(t) = \exp(-t/\tau_\varepsilon)$, $\varphi(t) = \exp(-t/\tau_\sigma)$ lead to the well-known relations for a standard linear solid

$$J' = J_\infty + \Delta J\,(1+\omega^2\tau_\sigma^2)^{-1},\; J'' = \Delta J\;\omega\tau_\sigma\;(1+\omega^2\tau_\sigma^2)^{-1}, \tag{7}$$

$$\operatorname{tg}\delta = \Delta J\omega\tau_\sigma\,(J_0 + J_\infty\omega^2\tau_\sigma^2)^{-1}, \tag{8}$$

$$(\Delta J = J_0 - J_\infty,\ \Delta E = E_\infty - E_0).$$

Here, J_0 is the relaxed and J_∞ the unrelaxed value of the compliance; τ_σ is the delay time, which is related with the relaxation time τ_ε by the expression

$$\tau_\varepsilon/\tau_\sigma = J_\infty/J_0 = E_0/E_\infty.$$

Expression (8) describes the peak of the relaxation internal friction, which reaches a maximum at

$$\omega\tau_\sigma\sqrt{J_\infty/J_0} = 1.$$

We note that the same formulas are easily obtained from the following rheological equation:

$$J_0\,(\sigma + \tau_\varepsilon\, d\sigma/dt) = \varepsilon + \tau_\sigma\, d\varepsilon/dt, \tag{9}$$

if we assume the harmonic variation of σ and ε with time.

2. An example of a kernel with a singularity at the instant of loading is the aftereffect kernel proposed by Duffing [5]

$$\varphi(t) = t^{\gamma-1} \qquad (0<\gamma<1). \tag{10}$$

Here and in what follows

$$\varkappa = \frac{\Delta J}{J_\infty\tau_\sigma^{\gamma}} = \frac{\Delta E}{E_\infty\tau_\varepsilon^{\gamma}}, \qquad \frac{J_\infty}{J_0} = \frac{E_0}{E_\infty} = \left(\frac{\tau_\varepsilon}{\tau_\sigma}\right)^{\gamma}. \tag{11}$$

Substituting (10) and (11) into (3) and going over to (5), we obtain the following values for the real and imaginary parts of the compliance:

$$J' = J_\infty + \Delta J\Gamma(\gamma)\,(\omega\tau_\sigma)^{-\gamma}\cos\psi,$$
$$J'' = \Delta J\,\Gamma(\gamma)(\omega\tau_\sigma)^{-\gamma}\sin\psi \qquad (\psi = {}^1/_2\pi\gamma). \tag{12}$$

The mechanical loss factor

$$\operatorname{tg}\delta = \frac{\Delta J\Gamma(\gamma)\sin\psi}{J_\infty(\omega\tau_\sigma)^{\gamma} + \Delta J\Gamma(\gamma)\cos\psi}. \tag{13}$$

The Duffing aftereffect kernel is equivalent to the rheological equation with fractional differentiation with respect to time [6]

$$\Gamma(\gamma)\,\Delta J\sigma + J_0\tau_\varepsilon^{\gamma} d^{\gamma}\sigma/dt^{\gamma} = \tau_\sigma^{\gamma} d^{\gamma}\varepsilon/dt^{\gamma}, \tag{14}$$

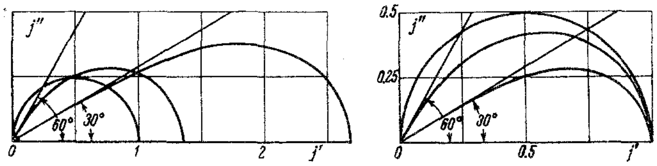
where γ is the order of the fractional derivative. Correct to the constant, Eq. (4) is the Maxwell model with fractional derivatives and, as $\gamma \to 1$, $J_0 \to \infty$, is transformed to the usual Maxwell rheological equation.

Equations (12)-(14) contain the gamma function $\Gamma(\gamma)$, which we can eliminate by taking the kernel in the Abel form

$$\varphi(t) = t^{\gamma-1}/\Gamma(\gamma).$$

3. The aftereffect kernel proposed by Rzhanitsyn [7] combines the properties of both the kernels considered above:

$$\varphi(t) = t^{\gamma-1}\exp(-t/\tau_\sigma) \qquad (0<\gamma\leqslant 1). \tag{15}$$



a b

Fig. 1

When $\gamma = 1$ we obtain the standard linear solid, and as $\tau_\sigma \to \infty$ the Duffing kernel.

Using kernel (15) in (3) and (5), we obtain

$$J' = J_\infty + \Delta J\,(1 + \omega^2\tau_\sigma^2)^{-1/2\gamma}\,\Gamma(\gamma)\cos\psi\,, \tag{16}$$

$$J'' = \Delta J\,(1 + \omega^2\tau_\sigma^2)^{-1/2\gamma}\,\Gamma(\gamma)\sin\psi\,,\quad \psi = \gamma\ \mathrm{arc\,tg}\,(\omega\tau_\sigma). \tag{17}$$

The mechanical loss factor

$$\mathrm{tg}\,\delta = \frac{\Delta J\Gamma(\gamma)\sin\psi}{J_\infty(1 + \omega^2\tau_\sigma^2)^{1/2\gamma} + \Delta J\Gamma(\gamma)\cos\psi}. \tag{18}$$

Figure 1a is the vector diagram of the complex compliance in relative units,

$$j'' = j''(j'), \quad j' \equiv (J' - J_\infty)/\Delta J, \quad j'' = J''/\Delta J.$$

We have taken the quantity γ as a parameter. As may be seen from Fig. 1, the curves are a synthesis of an arc of a circle ($\gamma = 1$, standard linear solid) and a straight line leaving the origin at an angle $\pi\gamma/2$ (Duffing and Abel kernels). Obviously, as the parameter γ decreases, the straight-line segment of the curve increases.

It is important to note that the presence of the gamma function $\Gamma(\gamma)$ in (16) and (17) leads to the curves intersecting the axis of abscissas at different points, namely, at the joint $j' = \Gamma(\gamma)$. In order to describe the relaxation at internal friction for a given modulus defect the curves in the vector diagram must start from the same points, and therefore the Rzhanitsyn kernel should be taken in the form

$$\varphi(t) = (t^{\gamma-1}/\Gamma(\gamma))\exp(-t/\tau_\sigma)\,, \tag{19}$$

i.e., with a correction for the gamma function $\Gamma(\gamma)$. The situation is analogous to that which exists between the Duffing and Abel kernels. The choice of a kernel in form (19) makes it possible to eliminate $\Gamma(\gamma)$ from (16)–(18), i.e., to set $\Gamma(\gamma) = 1$. The corresponding diagram is given in Fig. 1b. The only difference from the previous diagram is that as $\omega\tau_\sigma \to 0$ all the curves converge on the same point $j' = (J' - J_\infty)/\Delta J = 1$. For kernel (19) it is possible to write the following equivalent rheological equation:

$$\varepsilon = J_\infty\sigma + \Delta J\,(1 + \tau_\sigma d/dt)^{-\gamma}\sigma. \tag{20}$$

In order to construct the frequency dependence of the internal friction tg δ it is necessary to specify the modulus defect (compliance) as a parameter. However, the energy losses are proportional to the phase-shifted part of the deformation (stress), and the internal

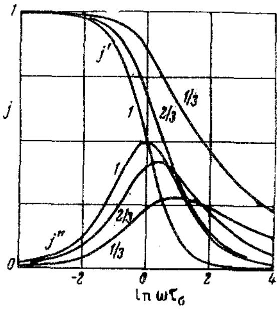


Fig. 2

friction curve is close to the curve representing the frequency dependence of the quantities J" or E". Therefore in obtaining qualitative information about the dissipative properties of the media it is possible to confine oneself to an investigation of the frequency dependence of the imaginary part of the compliance (modulus). The in-phase components J" and E" are the dynamic characteristics of

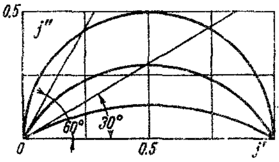


Fig. 3

the corresponding quantities. The relaxation time τ_ε and the delay time τ_σ are the material characteristics, which usually depend exponentially on temperature but do not depend on the frequency ω.

Figure 2 illustrates the frequency dependence of the real j' and imaginary j" components of the compliance. It is clear from Fig. 2 that a decrease in the parameter γ lowers the peak of the quantity j" and displaces it in the direction of higher frequencies as compared with the peak for a standard linear solid, which has a maximum of 0.5 at $\omega\tau_\sigma = 1$. A decrease in γ leads to a smoother variation of the dynamic compliance. Thus, the parameter γ characterizes the broadening and displacement of the retardation (relaxation) spectrum.

4. We will consider as kernels the exponential-fractional functions proposed by Rabotnov [8]. These kernels are convenient in that their resolvents are exponential-fractional functions of the same order.

For example, taking the aftereffect kernel

$$\varphi(t) = t^{\gamma-1}\sum_{n=0}^{\infty}\frac{(-1)^n (t/\tau_\sigma)^{\gamma n}}{\Gamma[\gamma(n+1)]} \tag{21}$$

and using Eqs. (4) we easily find that its resolvent is the relaxation kernel

$$f(t) = t^{\gamma-1}\sum_{n=0}^{\infty}\frac{(-1)^n (t/\tau_\varepsilon)^{\gamma n}}{\Gamma[\gamma(n+1)]}. \tag{22}$$

We note in passing that kernels (21) and (22) are equivalent to the rheological equation of a standard linear solid with fractional derivatives with respect to time

$$J_0(\sigma + \tau_\varepsilon^\gamma d^\gamma\sigma/dt^\gamma) = \varepsilon + \tau_\sigma^\gamma d^\gamma\varepsilon/dt^\gamma. \tag{23}$$

Using (21)–(23), we easily obtain

$$J' = J_\infty + \Delta J\frac{(\omega\tau_\sigma)^{-\gamma} + \cos\psi}{(\omega\tau_\sigma)^{\gamma} + (\omega\tau_\sigma)^{-\gamma} + 2\cos\psi}, \tag{24}$$

$$J'' = \Delta J\frac{\sin\psi}{(\omega\tau_\sigma)^{\gamma} + (\omega\tau_\sigma)^{-\gamma} + 2\cos\psi} \quad (\psi = {}^1/_2\pi\gamma)\,, \tag{25}$$

$$\mathrm{tg}\,\delta = \frac{\Delta J\sin\psi}{J_0(\omega\tau_\sigma)^{-\gamma} + J_\infty(\omega\tau_\sigma)^{\gamma} + (J_0 + J_\infty)\cos\psi}. \tag{26}$$

It is worth noting the total symmetry of Eqs. (21)–(26), which at $\gamma = 1$ go over into the ordinary relations for a standard linear solid.

In Fig. 3 we have plotted the $j'' = j''(j')$, diagram for various γ. This corresponds to Cole-Cole circle diagrams [9] with central angle $\gamma\pi$. The radius of each circle r is given by

$$r = {}^1/_2\ \mathrm{cosec}\,({}^1/_2\,\pi\gamma)\,, \tag{27}$$

i.e., the radius of the circle diagram is determined only by the parameter γ.

Then it is easy to establish that the angle $\psi = \pi\gamma/2$ determines the slope of the tangent to each arc relative to the axis of abscissas at the point 0 and 1. The tangents themselves correspond to the Abel kernel.

Thus, knowing γ, we can draw the vector diagram for Rabotnov kernels without additional computations.

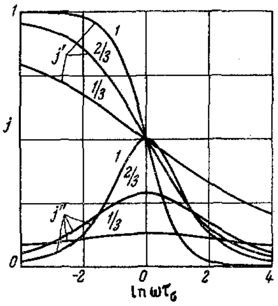


Fig. 4

It is clear from Fig. 4, which shows the frequency dependence of the real and imaginary parts of the compliance, that the parameter γ leads to broadening of the retardation (relaxation) spectrum.

More detailed information on the use of exponential-fractional kernels to describe internal friction of the relaxational type is given in [10].

Thus, the examples considered show that the singularity of kernels of the type $(t-\tau)^{\gamma-1}$ $(0 < \gamma \leq 1)$ determines the angle $\psi = \pi\gamma/2$ at which as $\omega \to \infty$ the curve of the vector diagram intersects the axis of abscissas along which are plotted real values of the compliance (modulus). At $\gamma = 1$ there is no singularity and the intersection is at right angles.

This is particularly apparent when we investigate the so-called background, that is, the sharp increase of internal friction tg δ with decrease in frequency (increase in temperature). From the phenomenological standpoint, the background is attributable to the total relaxation of the elastic modulus, which is reality is possible only for the shear modulus. Therefore, when the above equations are used describe the background, the elastic modulus should be interpreted as the shear modulus μ, which relaxes completely, i.e., $\mu_0 = 0$. In this case the Abel, Rzhanitsyn (19), and Rabotnov kernels lead to the same equation for the mechanical loss factor:

$$\mathrm{tg}\,\delta = [(\omega\tau_\varepsilon)^\gamma + \cos\psi]^{-1}\sin\psi, \qquad (28)$$

which at $\gamma = 1$ goes over into the well-known relation tg $\delta = 1/\omega\tau_\varepsilon$, obtained from the ordinary Maxwell equation.

It should be emphasized that the considerations leading to Eq. (28) indicate that the nature of the background has a purely relaxational character unrelated with any retardation process.

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